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A Banach Algebra Version of the Sato Grassmannian and Commutative Rings of Differential Operators

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Abstract We show that commutative rings of formal pseudodifferential operators can be conjugated as subrings in noncommutative Banach algebras of operators in the presence of certain eigenfunctions. Techniques involve those of the Sato Grassmannian as used in the study of the KP hierarchy as well as the geometry of an infinite dimensional Stiefel bundle with structure modeled on such Banach algebras. Generalizations of this procedure are also considered.

Key words semigroup · Fredholm operator · Sato Grassmannian · KP hierarchy · Burchnall–Chaundy ring · iterated Laurent series.

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1 Introduction

The importance of Fredholm structures has for some time been recognized in the theory of holomorphic operator-valued functions and extensive work has been accomplished in this area as seen, for instance, in [14–17, 37] (see also the many references cited therein). Related studies involve the Grassmann models of systems-control theory as studied in e.g. [8, 18, 19]. In the background stands another substantial amount of research concerning the structure of spaces of projections in Banach algebras and their associated manifolds, providing essential techniques for operator-theorists; references to this subject include [3, 7, 10–12, 30]. Grassmannians with Fredholm structure also feature in the Riemann–Hilbert and elliptic transmission problems as studied in [6, 21] in which C*-algebra techniques are also incorporated.

On the other hand Grassmann models in both finite and infinite dimensions are essential objects of study in algebraic geometry, integrable systems, and in other areas of mathematics. This paper provides some new insight into the operator-theoretic approach and techniques of integrable systems along a commutative to a noncommutative route. The initial commutative objects concern the Burchnall–Chaundy ring of formal pseudodifferential operators which plays a significant role in the theory of the Kadomtsev–Petviashvili (KP) hierarchy and related integrable systems. A fundamental work of Sato [34] relates such rings of operators to the Korteweg–de Vries (K–dV) equation [34] where solutions of the latter correspond to points in an infinite dimensional Grassmannian. The interpretation and further implications of the Sato Grassmannian were unfolded with technical depth in [35] (which [32] extends to vector Grassmannians). Because of its analytic (Fredholm) structure, the Grassmann model of [35] has further provided a valuable technique in studying many types of integrable systems and infinite-dimensional Lie algebras (see e.g. [1, 25, 27, 31]). An operator-theoretic account on the fringe of these topics can be found in [18] where, among various applications, it is observed that solution flows of equations such as the K–dV arise on shift-invariant subspaces of certain Banach spaces.

The range of ideas at stake suggested a further technical calibration of the prevailing geometric correspondences in a relatively new direction. Specifically, to determine the role that operator algebras may play in order to cover the yet-unclassified algebras of partial differential operators that are (in principle) linked to the algebraic-geometric study of multidimensional spectral varieties and the moduli spaces of vector bundles. Whereas the latter almost exclusively concern 'formal solutions,' it is interesting to adopt a function-theoretic framework that may create a bridge towards the burgeoning field of noncommutative geometry. In quest of such a connection we start here by examining the interplay between the commutative Burchnall–Chaundy rings and the generally noncommutative nature of the corresponding algebras of operators over certain Hilbert modules. The theme and main results of this paper perhaps best encapsulate one part of the prospective development by realizing the former as conjugated (with some dexterity) into subrings of the latter. It is somewhat in the spirit of how the infinite matrices of [34] and differential operators of the Burchnall–Chaundy ring are in a 'boson-fermion correspondence,' thanks to implementing the Baker function, a function which from the operator-theoretic point of view seems to have been curiously overlooked. A particular aspect of our approach concerns applying the general construction of the

Stiefel bundle concept as definable over a topological algebra (in fact, over a possibly noncommutative ring). This construction has featured in a series of papers [9–13](see also the references therein) and can be adapted to various classes and categories of differentiability. More specifically, it is within this setting that we interpret the relevant conjugation principles of [32, 35] where Baker functions are employed. Certain generalizations of [32, 35] to several variables are studied in papers such as [28, 29] (see also the survey article [33] and references therein). Although this latter work is essentially algebraic-geometric, we outline how the analogous conjugation results can be seen within the proposed setting of operator theory.

2 Algebraic Preliminaries

2.1 The Space of Idempotents $P(A)$ and the Grassmannian $\mathrm{Gr}(A)$

Our references for this section are mainly [10, 12] (and references therein).

To commence, let A be a monoidal (multiplicative) semigroup with group of units denoted by $G(A)$. Let

$$P(A) := \{p \in A \ : \ p^2 = p\}, \tag{2.1}$$

that is, $P(A)$ is the set of idempotent elements in A (for suitable A, we can regard elements of $P(A)$ as projections). Recall that the right Green's relation is $p\mathcal{R}q$ if and only if $pA = qA$ for $p, q \in A$.

Let $\mathrm{Gr}(A) = P(A)/\mathcal{R}$ be the set of equivalence classes in $P(A)$ under $\mathcal{R}$. As the set of such equivalence classes, $\mathrm{Gr}(A)$ will be called *the Grassmannian of* A. Relative to a given topology on A, $\mathrm{Gr}(A)$ is a space with the quotient topology resulting from the natural quotient map

$$\Pi \ : P(A) \longrightarrow \mathrm{Gr}(A). \tag{2.2}$$

Let $h : A \longrightarrow B$ be a semigroup homomorphism. Then it is straightforward to see that the diagram below is commutative:

$$\begin{array}{ccc} P(A) & \xrightarrow{P(h)} & P(B) \\ \Pi \downarrow & & \downarrow \Pi \\ \mathrm{Gr}(A) & \xrightarrow{\mathrm{Gr}(h)} & \mathrm{Gr}(B) \end{array} \tag{2.3}$$

2.2 The Space of Partial Isomorphisms $W(A)$

Definition 2.1 We say that $u \in A$ is a *partial isomorphism* if there exists a $v \in A$ such that $uvu = u$ and $vuv = v$, in which case we call v a *relative inverse* (or *pseudoinverse*) for u. In general such a relative inverse is not unique. We take $W(A)$ to denote the set (or space, if A has a topology) of all partial isomorphisms of A.

If $u \in W(A)$ has a relative inverse v, then clearly $v \in W(A)$ with relative inverse u, and it is easy to see that both vu and uv belong to $P(A)$. Although v is not uniquely determined by u alone, it is uniquely determined once u, vu and uv are all specified [10].

If $p \in P(A)$, then we take $W(p, A) \subset W(A)$ to denote the subspace of all partial isomorphisms u in A having a relative inverse v satisfying $vu = p$. Likewise, $W(A, q)$ denotes the subspace of all partial isomorphisms u in A having a relative inverse v satisfying $uv = q$, so that we have $W(A, q) = W(q, A^{op})$. Now for $p, q \in P(A)$, we set

$$\begin{aligned} W(p, A, q) &= W(p, A) \cap W(A, q) \\ &= \{ u \in qAp : \exists\, v \in pAq\,,\ vu = p \text{ and } uv = q \}. \end{aligned} \tag{2.4}$$

Remark 1 A partial isomorphism is equivalently a *pseudoregular element*, a nomenclature sometimes used in the literature.

2.3 The Space of Proper Partial Isomorphisms $V(A)$

Recall that two elements $x, y \in A$ are *similar* if x and y are in the same orbit under the inner automorphic action $*$ of $G(A)$ on A . For $p \in P(A)$, we say that the orbit of p under the inner automorphic action is *the similarity class of* p and denote the latter by $\mathrm{Sim}(p, A)$, whereby it follows that $\mathrm{Sim}(p, A) = G(A) * p$.

Definition 2.2 Let $u \in W(A)$. We call u a *proper partial isomorphism* if for some $W(p, A, q)$, we have $u \in W(p, A, q)$ where p and q are similar. We take $V(A)$ to denote the space of all proper partial isomorphisms of A.

Observe that $G(A)V(A)$ and $V(A)G(A)$ are both subsets of $V(A)$. In the following we set $G(p) = G(pAp)$.

2.4 The Spaces $V(p, A)$ and $\mathrm{Gr}(p, A)$

If $p \in P(A)$, then we denote by $V(p, A)$ the space of all proper partial isomorphisms of A having a relative inverse $v \in W(q, A, p)$ for some $q \in \mathrm{Sim}(p, A)$. With reference to (2.4) this condition is expressed by

$$V(p, A) := \bigcup_{q \in \mathrm{Sim}(p, A)} W(p, A, q). \tag{2.5}$$

Notice $V(p, A) \subset V(A) \cap W(p, A)$, but equality may not hold. Clearly, we have $G(A) \cdot p \subset V(p, A)$ and just as in [10] it can be shown that equality holds if A is a ring. The image of $\mathrm{Sim}(p, A)$ under the map Π defines the space $\mathrm{Gr}(p, A)$ viewed as the Grassmannian naturally associated to $V(p, A)$.

For a given unital semigroup homomorphism $h : A \longrightarrow B$, there is a restriction of (2.3) to a commutative diagram:

$$
\begin{array}{ccc}
V(p, A) & \xrightarrow{V(p,h)} & V(q, B) \\
\Big\downarrow{\scriptstyle \Pi_A} & & \Big\downarrow{\scriptstyle \Pi_B} \\
\mathrm{Gr}(p, A) & \xrightarrow{\mathrm{Gr}(p,h)} & \mathrm{Gr}(q, B)
\end{array}
\tag{2.6}
$$

where for $p \in P(A)$, we have set $q = h(p) \in P(B)$. Observe that in the general semigroup setting, $V(p, A)$ properly contains $G(A)p$. In fact, if $p \in P(A)$, then $V(p, A) = G(A)G(pAp)$ (see [12] Lemma 2.3.1).

Let $H(p)$ denote the isotropy subgroup for this left-multiplication. We have then the coset space representation $\mathrm{Gr}(p, A) = G(A)/G(\Pi(p))$ where $G(\Pi(p))$ denotes the isotropy subgroup of $\Pi(p)$. Then there is the inclusion of subgroups $H(p) \subset G(\Pi(p)) \subset G(A)$, resulting in a fibering $V(p, A) \longrightarrow \mathrm{Gr}(p, A)$ given by the exact sequence

$$G(\Pi(p))/H(p) \hookrightarrow G(A)/H(p) \longrightarrow G(A)/G(\Pi(p)), \tag{2.7}$$

generalizing the well-known *Stiefel bundle* construction in finite dimensions.

Remark 2 It is clear that the above constructions are purely algebraic and follow from the development of [10] Sections 3–7 and [12] Sections 1–2. For A a Banach algebra, the map $\Pi : P(A) \longrightarrow \mathrm{Gr}(A)$ is open, a fact also deduced in [30] which considers the structure of $\mathrm{Gr}(A)$ but from a different viewpoint.

2.5 Specialization to a Banachable Algebra

Most often A will be taken to be a subsemigroup of a ring of operators, or, of the partial isomorphisms of a topological algebra [36]. The latter include Fréchet algebras (such as the Ψ-algebras considered in [16]), as well as the Banachable algebras, a class of topological algebras whose underlying vector space is a Banach space. Clearly, any Banach algebra satisfies this latter property. Other examples include the Jordan–Lie algebras whose underlying vector spaces are Banach spaces (the JLB-algebras studied in [24]).

As shown in [10], the Banachable assumption on A is necessary to endow a Banach analytic manifold structure on the space in question. In this case, $V(p, A)$ may be viewed as a *space of framings* for elements of $\mathrm{Gr}(p, A)$. In particular, we recall from [10] Section 7 that on setting $G(p) = G(pAp)$, we have $\mathrm{Gr}(p, A) = V(p, A)/G(p)$, and there is a (locally trivial) principal fibration

$$G(p) \hookrightarrow V(p, A) \longrightarrow \mathrm{Gr}(p, A). \tag{2.8}$$

Thus the homogeneous fibration (2.7) becomes an analytic locally trivial fibration for the relevant category of differentiability (the reader may wish to consult [11] to see how the situation looks in the holomorphic category for instance). Also, on setting

$\hat{p} = 1 - p$, there is the following expression for the (pointwise) decomposition of tangent spaces:

$$T\mathrm{Gr}(p, A) = pA\hat{p} + \hat{p}Ap. \tag{2.9}$$

2.6 The Spatial Correspondence

Returning to the general case, if $\mathcal{A}$ is a given topological algebra and E is some $\mathcal{A}$-module, then $A = \mathcal{L}_{\mathcal{A}}(E)$ could be taken to be the ring of $\mathcal{A}$-linear transformations of E. An example is when E is a complex Banach space and $A = \mathcal{L}(E)$ is the Banach algebra of bounded linear operators on E. In order to understand the interface between spaces such as $\mathrm{Gr}(p, A)$ and the usual Grassmannians of subspaces (of a vector space E), we will describe a 'spatial correspondence.'

Given a topological algebra $\mathcal{A}$, suppose E is an $\mathcal{A}$-module admitting a decomposition

$$E = F \oplus F^c\,, \qquad F \cap F^c = \{0\}, \tag{2.10}$$

where F, F^c are closed subspaces of E . We have already noted $A = \mathcal{L}(E)$ as the ring of linear transformations of E. Here $p \in P(E) = P(\mathcal{L}(E))$ is chosen such that $F = p(E)$, and consequently $\mathrm{Gr}(A)$ consists of all such closed splitting subspaces. The assignment of pairs $(p, \mathcal{L}(E)) \mapsto (F, E)$, is called a *spatial correspondence*, and so leads to a commutative diagram

$$\begin{array}{ccc} V(p, \mathcal{L}(E)) & \xrightarrow{\ \varphi\ } & V(p, E) \\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \Pi} \\ \mathrm{Gr}(p, \mathcal{L}(E)) & \xrightarrow{\ =\ } & \mathrm{Gr}(F, E) \end{array} \tag{2.11}$$

where $V(p, E)$ consists of linear homomorphisms of $F = p(E)$ onto a closed splitting subspace of E similar to F. In particular, the points of $\mathrm{Gr}(p, \mathcal{L}(E))$ are in a 1 : 1 correspondence with those of $\mathrm{Gr}(F, E)$.

Our description so far reveals a framework for considering just about all species of Grassmannians employed in geometry and analysis (including, of course, the usual finite-dimensional ones). We will proceed to make the necessary specializations as the situation arises.

Example 2.1 The following relates to the Grassmann model of [34]. Let $\mathbb{C}^\infty = \mathbb{C}^{(\mathbb{N})}$ denote the space of all finite sequences $\{z_k\}$, $z_k \in \mathbb{C}$, and $G_k(\mathbb{C}^\infty)$ the Grassmannian of all k-dimensional linear subspaces of $\mathbb{C}^\infty$. The Stiefel manifold $V_k(\mathbb{C}^\infty)$ is the manifold of all k-frames, that is, the set of all injective linear maps $\mathbb{C}^k \longrightarrow \mathbb{C}^\infty$. Specifically

$$V_k(\mathbb{C}^\infty) = \{\Lambda \in \mathcal{L}(\mathbb{C}^k, \mathbb{C}^\infty) : \Lambda^{\mathrm{t}} \circ \Lambda \in \mathrm{GL}(k, \mathbb{C})\}. \tag{2.12}$$

Following e.g. [22] (Theorem 47.5), we observe that an embedding $\mathbb{C}^k \longrightarrow \mathbb{C}^\infty$, induces in the inductive limit, the embeddings

$$V_k(\mathbb{C}^\infty) = \varinjlim_N V_k(\mathbb{C}^N),$$

$$G_k(\mathbb{C}^\infty) = \varinjlim_N G_k(\mathbb{C}^N). \tag{2.13}$$

The situation considered in [34] (c.f. [35]) is essentially that of $N = 2n$ and $k = n$ (for some n). We will recall several features of this work in the following section(s).

3 Fredholm Grassmannians

Here we will consider several important restrictions of the fibration $V(p, A) \longrightarrow \mathrm{Gr}(p, A)$ when A is a certain Banach algebra. To proceed, we denote by $\mathcal{L}(E, F)$, the Banach space of bounded linear operators between complex Banach spaces E and F, and when $E = F$, $\mathcal{L}(E)$ denotes the resulting complex Banach algebra. The space of Fredholm operators when definable on a space Z, is denoted by $\mathrm{Fred}(Z)$.

3.1 The Schatten Classes

We refer to [25, 31] for the following facts which are relevant to our situation. Given a pair of Hilbert spaces (H_1, H_2) and $1 \leq \alpha < \infty$, we denote by $\mathcal{L}_\alpha(H_1, H_2)$ the *α–Schatten ideal* in $\mathcal{L}(H_1, H_2)$ of adjoinable linear operators $T : H_1 \to H_2$, satisfying

$$\| T\|_\alpha^\alpha = \mathrm{Tr}(T^* T)^{\frac{\alpha}{2}} < \infty. \tag{3.1}$$

The $\mathcal{L}_\alpha(H_1, H_2)$ are Banach spaces and form an increasing chain of ideals within the compact operators denoted by $\mathcal{L}_\infty(H_1, H_2)$. For instance, the case $\alpha = 1$ corresponds to trace class operators when $H_1 = H_2$, and $\alpha = 2$ corresponds to the Hilbert–Schmidt operators (whereby $\mathcal{L}_2(H_1)$ is a Hilbert space).

Suppose H is a complex separable Hilbert space admitting a decomposition $H = H_1 \oplus H_2$ of the type (2.10), where H_1 and H_2 are infinite dimensional closed subspaces with $H_1 \cap H_2 = \{0\}$. For $s \geq 1$, consider the group of invertible bounded operators defined by

$$\mathrm{GL}_s = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\}, \tag{3.2}$$

where $a \in \mathrm{Fred}(H_1)$, $b \in \mathcal{L}_{2s}(H_2, H_1)$, $c \in \mathcal{L}_{2s}(H_1, H_2)$, $d \in \mathrm{Fred}(H_2)$. We define GL_0 to consist of those elements for which b and c are finite rank operators. With suitable metric topologies, each GL_s is a complex analytic Banach Lie group modeled on the appropriate Banach space, and there is a chain of embeddings $\mathrm{GL}_0 \subset \mathrm{GL}_1 \subset \cdots \subset \mathrm{GL}_\infty$, where for $s' \leq s$, each $\mathrm{GL}_{s'}$ is dense in GL_s. There is also a corresponding chain of embeddings of Banach analytic Lie groups given by $\mathrm{GL}^0 \subset \mathrm{GL}^1 \subset \cdots \subset \mathrm{GL}^\infty$, where for $s \geq 1$, GL^s consists of all invertible linear operators on H of the form $T - 1 \in \mathcal{L}_s(H_1, H_2)$.

Let Δ_s be the subgroup of block triangular operators in GL_s where $c = 0$. This leads to defining the Grassmannian $\mathrm{Gr}_s(H_1, H) = \mathrm{GL}_s/\Delta_s$ for which Δ_s is the stabiliser of the subspace H_1. Elements of $\mathrm{Gr}_s(H_1, H)$ may be viewed as closed subspaces $W = g \cdot H_1$, for some $g \in \mathrm{GL}_s$ as in (3.2), which split H and which are similar to H_1. In this way, $\mathrm{Gr}_s(H_1, H)$ can be seen to be a Banach manifold modeled on the Banach space $\mathcal{L}_{2s}(H_1, H_2)$, and is thus an example of the Banach manifold $\mathrm{Gr}(F, E)$.

For elements $W \in \mathrm{Gr}_s(H_1, H)$, there are certain bases called *admissible bases* which, loosely speaking, allow us to make sense of a determinant det W, in part thanks to the group GL^s. The relevant space $V(p, A)$ is the Stiefel manifold of admissible bases denoted by $V_s(H_1, H)$, for which there is a principal GL^s-bundle

$$\mathrm{GL}^s \hookrightarrow V_s(H_1, H) \longrightarrow \mathrm{Gr}_s(H_1, H), \tag{3.3}$$

being the restriction to such bases of the locally trivial principal $\mathrm{GL}(H)$-bundle given by (2.8).

3.2 The Banach Algebra $A = \mathcal{L}_J(H)$

Let H be a separable complex Hilbert space admitting an orthogonal direct sum decomposition $H = H_+ \oplus H_-$, where the $H_\pm$ are closed subspaces, for which the decomposition is specified by a unitary operator $J : H \to H$ such that $J|H_\pm = \pm 1$. In the context of the preceding example, we now consider (without loss of generality) the case $s = 1$ and set $\mathrm{Gr}_1(H_+, H) = \widehat{\mathrm{Gr}}(H_+, H)$. Relative to $\widehat{\mathrm{Gr}}(H_+, H)$, we consider closed splitting subspaces W that are *commensurable with* H_+ (that is, for which $W \cap H_+$ has finite codimension in both W and H_+). Specifically, elements of $\widehat{\mathrm{Gr}}(H_+, H)$ consist of closed subspaces $W \subset H$ such that:

(1) The orthogonal projection $p_+ : W \longrightarrow H_+$ is Fredholm, and
(2) The orthogonal projection $p_- : W \longrightarrow H_-$ is Hilbert–Schmidt .

The subspaces $W \in \widehat{\mathrm{Gr}}(H_+, H)$ on which p_+ is an isomorphism, form a dense open subset of the latter which is called the *big cell.*

The relevant algebra to consider for these Grassmannians is $A = \mathcal{L}_J(H)$, the Banach algebra of bounded linear operators $T : H \to H$ such that $[J, T]$ is a Hilbert–Schmidt operator. The relevant norm $\| \ \|_J$ is defined by

$$\|T\|_J = \|T\| + \|[J, T]\|_2. \tag{3.4}$$

Together with the topology induced by $\| \ \|_J$, the group of units $G(A)$ is a complex Banach Lie group. Now the subgroup of unitaries $U(A) \subset G(A)$ acts transitively on $\widehat{\mathrm{Gr}}(H_+, H)$, and consequently $G(A)$ is seen to be identifiable with GL_1 following (3.2) (see [31] Sections 6.2, 7.1).

For our purposes we will often deal directly with the Banach algebra A . For the appropriate choice of $p \in P(A) \subset A$ such that $p \in \mathrm{Sim}(p_+, A)$, we will identify $\mathrm{Gr}(p, A)$ with $\widehat{\mathrm{Gr}}(H_+, H)$ via the spatial correspondence. Further, the Banach Lie group $G(p) = G(pAp)$ is identified with GL^1, and elements $w = \{w_i\} \in V(p, A)$, are viewed as admissible bases for W as above, and are preserved under $G(p) = \mathrm{GL}^1$. In view of (3.3) for the case $s = 1$, we have

$$\mathrm{Gr}(p, A) = V(p, A)/G(p) = G(A)/G(\Pi(p)) \,, \tag{3.5}$$

whereby $\mathrm{Gr}(p, A)$ is a complex Hilbert manifold modeled on $\mathcal{L}_2(H_+, H_-)$. We also have the universal bundle $\mathcal{E}(p, A) \longrightarrow \mathrm{Gr}(p, A)$ which is the holomorphic Hilbert bundle associated to the principal bundle $V(p, A) \longrightarrow \mathrm{Gr}(p, A)$ (a more general situation is considered in [11]). For tangent spaces (pointwise), we have the complex type decomposition

$$T^{\mathbb{C}}\mathrm{Gr}(p, A) = T'\mathrm{Gr}(p, A) \oplus T''\mathrm{Gr}(p, A), \tag{3.6}$$

where for $W = p(H) \in \mathrm{Gr}(p, A)$ we deduce from (2.9):

$$\begin{aligned} T'_W\mathrm{Gr}(p, A) &\cong \mathrm{Hom}_{\mathbb{C}}(W, W^{\perp}) \cong pA\hat{p}\,, \\ T''_W\mathrm{Gr}(p, A) &\cong \mathrm{Hom}_{\mathbb{C}}(W^{\perp}, W) \cong \hat{p}Ap\,. \end{aligned} \tag{3.7}$$

4 Subrings of Formal Pseudodifferential Operators

In this section we will start on the 'commutative' side of the picture by considering certain rings of pseudodifferential operators as studied in [32, 35].

4.1 The Burchnall–Chaundy Ring of Formal Pseudodifferential Operators

Let $\mathbb{B}$ denote the algebra of analytic functions $U \longrightarrow \mathbb{C}$ where U is a connected open neighbourhood of the origin in $\mathbb{C}$. The (generally noncommutative) algebra $\mathbb{B}[\partial]$ of linear differential operators with coefficients in $\mathbb{B}$, consists of expressions

$$\sum_{i=0}^{N} a_i\,\partial^i\,,\ (a_i \in \mathbb{B}\,,\ \text{ for some } N \in \mathbb{Z}). \tag{4.1}$$

Here $\partial \equiv \partial/\partial x$ and the a_i can be regarded as multiplication operators for which multiplication is defined by

$$[\partial, a] = \partial a - a\partial = d(a) \equiv \partial a/\partial x. \tag{4.2}$$

Let $\mathbb{A}$ denote a commutative subalgebra of $\mathbb{B}[\partial]$. The *rank* of $\mathbb{A}$ is defined to be the greatest common divisor of all orders of the operators in $\mathbb{A}$. For instance, given an operator of the form

$$L_0 = \partial^r + u_{r-2}\,\partial^{r-2} + \ldots u_1\,\partial + u_0, \tag{4.3}$$

the algebra $\mathbb{C}[L_0]$ has rank equal to r.

Next we proceed to the algebra $\mathbb{B}[\partial^{-1}]$ of formal pseudodifferential operators with coefficients in $\mathbb{B}$. This algebra is obtained from $\mathbb{B}[\partial]$ by formally inverting the operator ∂ . Thus $P \in \mathbb{B}[\partial^{-1}]$ may be written as

$$P = \sum_{i>-\infty}^{N} a_i\,\partial^i. \tag{4.4}$$

The operator P admits a decomposition $P = P_+ + P_-$, where

$$P_+ = \sum_{i=0}^{N} a_i\,\partial^i\,,\ P_- = \sum_{i>-\infty}^{-1} a_i\,\partial^i. \tag{4.5}$$

4.2 The Formal Baker Function ψ_W

Recall that the nth generalized K–dV-hierarchy consists of all evolution operators for $n-1$ unknown functions $u_0(x,t),\ldots,u_{n-2}(x,t)$ that can be expressed as

$$\frac{\partial L}{\partial t} = [P, L], \tag{4.6}$$

where $L \in \mathbb{B}[\partial^{-1}]$ is an nth order differential operator

$$L = \partial^n + u_{n-2}\,\partial^{n-2} + \ldots u_1\,\partial + u_0, \tag{4.7}$$

and P is a differential operator for which ord $[P, L] \leq (n-2)$. In order to study the evolution of eigenfunctions of L via comparison with the constant operators ∂^n, we want to find an operator K conjugating L such that $K(L)K^{-1} = \partial^n$. So if ψ_0 is an eigenfunction of ∂^n, then $\psi = K\psi_0$ will consequently be an eigenfunction of L . Following [35] Section 4, there exists such an operator $K \in \mathbb{B}[\partial^{-1}]$ given by

$$K = 1 + \sum_{i=1}^{\infty} a_i(x)\,\partial^{-i}, \tag{4.8}$$

which is determined up to right multiplication by a constant coefficient operator of the form $1 + c_1\,\partial^{-1} + \ldots$ (noting that only constant coefficient operators commute with ∂^n).

Starting from the Hilbert space $H = L^2(S^1, \mathbb{C}^n)$, we recall the Banach algebra $A = \mathcal{L}_J(H)$ and the Fredholm Grassmanian $\widehat{\mathrm{Gr}}(H_+, H) = \mathrm{Gr}(p, A)$. We next consider an exclusive class of eigenfunctions obtained from the development of [35] Section 5. Let D be the closed unit disc and denote by Γ_+ the group of holomorphic maps $g : D \longrightarrow \mathbb{C}^*$, such that $g(0) = 1$. The group Γ_+ acts on $\widehat{\mathrm{Gr}}(H_+, H)$ via multiplication operators on H, and for each $W \in \widehat{\mathrm{Gr}}(H_+, H)$, we define a dense open subset of Γ_+ by

$$\Gamma_+^W = \{g \in \Gamma_+ : g^{-1}W \text{ is transverse to } H_-\}\,. \tag{4.9}$$

We also define Γ_- as the group of holomorphic maps of the form $g\left(\frac{1}{z}\right)$ where $g \in \Gamma_+$. It can be seen that Γ_- acts freely and transitively on $\widehat{\mathrm{Gr}}(H_+, H) = \mathrm{Gr}(p, A)$ (recall that $p \in \mathrm{Sim}(p_+, A)$) .

Now for a given $g \in \Gamma_+^W$ there exists a unique $h_g \in \Gamma_-$ determined by W, satisfying the relation $g^{-1}W = h_g H_+$, that is,

$$W = g h_g H_+. \tag{4.10}$$

Accordingly, there exists a function $\psi_W = \psi_W(x, z)$ given by

$$\psi_W = \psi_W(x, z) = g(z)\left(1 + \sum_{i=1}^{\infty} a_i(g)\,z^{-i}\right), \tag{4.11}$$

where the a_i are analytic functions on Γ_+^W extending to meromorphic functions on all of Γ_+ . The function ψ_W is the unique function of that type lying in $g^{-1}W$, and via orthogonal projection $p_+^g : g^{-1}W \longrightarrow H_+$, we have $\psi_W = (p_+^g)^{-1}(\mathbf{1})$. Further, the $g \in \Gamma_+$ can be expressed uniquely as

$$g(z) = \exp(xz + t_2 z^2 + t_3 z^3 + \cdots), \tag{4.12}$$

so that $\psi_W = \psi_W(x, z; t_2, t_3, \ldots)$.

We call ψ_W the *(formal) Baker function of the subspace* $W \in \widehat{\mathrm{Gr}}(H_+, H)$. Accordingly, there exists a unique differential operator P_n such that

$$P_n \psi_W = z^n \psi_W. \tag{4.13}$$

Furthermore, the operator P_n admits the conjugation property $P_n = K(\partial^n) K^{-1}$.

4.3 First Conjugation Result

In order to simplify the exposition, we will relax temporarily the dependence on the variables $(t_2, t_3, \ldots)$, and so $\psi_W = \psi_W(x, z)$ unless otherwise stated.

We start by considering the eigen-equation $L\psi = z^n \psi$ which admits a formal power series solution as determined by the Baker function

$$\psi = e^{xz} \left(1 + \sum_{i=1}^{\infty} a_i(x)\, z^{-i} \right), \tag{4.14}$$

which is unique up to multiplication with constant coefficients of the form $1 + c_1\, z^{-1} + \ldots$.

It will be sufficient to deal with the case $n = 1$, and take an operator L of order 1 in a commutative subring $\widehat{\mathbb{A}} \subset \mathbb{B}[\partial^{-1}]$, such that L is given by

$$L = \partial + \sum_{i>-\infty}^{-1} u_i\, \partial^i. \tag{4.15}$$

Using the elegant idea of [34], the correspondence '$\left(\frac{\partial}{\partial x}\right)^{-1} \leftrightarrow$ multiplication by z,' here realizes commutative subrings $\widehat{\mathbb{A}} \subset \mathbb{B}[\partial^{-1}]$ as subrings of $\mathbb{C}[[z]][z^{-1}]$. Thus for $H = L^2(S^1, \mathbb{C})$ and the Banach algebra $A = \mathcal{L}_J(H)$, we establish the following preliminary result which may be seen as an interpretation of [34, 35] in terms of the relevant algebras.

Theorem 4.1 *Given the Baker function ψ_W associated to a subspace $W \in \widehat{\mathrm{Gr}}(H_+, H)$, the ring $\widehat{\mathbb{A}} \subset \mathbb{B}[\partial^{-1}]$, conjugates as a subring into the Banach algebra $A = \mathcal{L}_J(H)$ up to constant coefficient operators.*

Proof We start with the subspace $W \in \widehat{\mathrm{Gr}}(H_+, H) = \mathrm{Gr}(p, A)$ given by (4.10) , and consider the assigned Baker function $\psi_W = \psi_W(x, z)$. We can assume that W belongs to the big cell, so under the orthogonal projection $p_+^g : W \longrightarrow H_+$, we have $\psi_W = (p_+^g)^{-1}(\mathbf{1})$. For $P \in \widehat{\mathbb{A}}$, $\partial = \frac{\partial}{\partial x}$, we assign $P\left(\frac{\partial}{\partial x}\right)$ to $f(z^{-1})$ via

$$P\Big(\frac{\partial}{\partial x}\Big)\psi_W = f(z^{-1})\psi_W. \tag{4.16}$$

The form of W given by (4.10) and the definition of ψ_W lead to $\psi_W = We^{zx}$. With L as in (4.15), the relevant eigenvalue problem of (4.13) $L\psi_W = z\psi_W$, thus corresponds to $\partial(e^{zx}) = ze^{zx}$.

Recall from (4.8), we have $K = 1 + \sum_{i=1}^{\infty} a_i(x)\, \partial^{-i}$, such that each $L \in \widehat{\mathbb{A}}$ satisfies the conjugation property $L = K(\partial)K^{-1}$. Comparing this with (4.10) and (4.11), we can write $K = h_g(\partial)$, for $h_g \in \Gamma_-$. Thus via the assignment $\partial \mapsto z^{-1}$, we obtain

$$L \mapsto h_g(z^{-1})\,(z^{-1})\,(h_g(z^{-1}))^{-1}. \tag{4.17}$$

Since the h_g was uniquely determined by the $W \in \mathrm{Gr}(p, A)$, the latter therefore determines the conjugation of $\widehat{\mathbb{A}}$ into a commutative subring of A. □

Theorem 4.1 is a theme on which we shall perform certain variations, but in stages of generality. Implicit in the proof of Theorem 4.1 is the fact that the KP-hierarchy is manifestly a sequence of deformations for $L = K(\partial)K^{-1}$, $K \in \widehat{\mathbb{A}}$, whereby the Baker (eigen)function $\psi_W = We^{zx}$ converts differentiation into multiplication. As shown explicitly in [2] (see also [32, 35]), the solution flow of $L\psi_W = z\psi_W$, arises on subspaces W for which the vectors

$$(\psi_W,\ \partial\psi_W, \ldots,\ \partial^{k-1}\psi_W, \ldots)|_{x=0}\ ,\ (\partial \equiv \partial/\partial x), \tag{4.18}$$

comprise a basis for W, the so-called ‘admissible bases’ which will be discussed at a later stage.

5 Grassmannians Over Hilbert C*-modules

5.1 Standard Operators on Hilbert C*-modules

Let us start by describing some basic properties of Hilbert modules over C*-algebras. We shall skip some of the usual elementary definitions since we will be restricting our discussion to a particular class; thus the reader is referred to e.g. [5, 23, 24] for a comprehensive account of the subject.

Consider a unital C*-algebra $\mathcal{A}$ and the standard (free countable dimensional) Hilbert module $H_{\mathcal{A}}$ over $\mathcal{A}$ as given by

$$H_{\mathcal{A}} = \{\{\zeta_i\}\ ,\ \zeta_i \in \mathcal{A}\ ,\ i \geq 1\ :\ \sum_{i=1}^{\infty} \zeta_i\zeta_i^* \in \mathcal{A}\,\} \cong \oplus\mathcal{A}_i, \tag{5.1}$$

where each $\mathcal{A}_i$ represents a copy of $\mathcal{A}$.

Example 5.1 Suppose H is a separable Hilbert space. We can form the algebraic tensor product $H \otimes_{\mathrm{alg}} \mathcal{A}$ on which there is an $\mathcal{A}$-valued inner product

$$\langle x \otimes \zeta\ ,\ y \otimes \eta\rangle\ =\ \langle x, y\rangle\ \zeta^*\eta\ , \qquad x, y \in H\ ,\ \zeta, \eta \in \mathcal{A}. \tag{5.2}$$

Thus $H \otimes_{\mathrm{alg}} \mathcal{A}$ becomes an inner product $\mathcal{A}$–module whose completion is denoted by $H \otimes \mathcal{A}$. Given an orthonormal basis for H, we have the following identification (unitary equivalence) given by $H \otimes \mathcal{A} \approx H_{\mathcal{A}}$ (see e.g. [23]).

There is also a natural $\mathcal{A}$-valued scalar product on $H_{\mathcal{A}}$ leading to classes of bounded linear operators on $H_{\mathcal{A}}$ (as studied in e.g. [26]). Let $A_0 = \mathcal{L}(H_{\mathcal{A}})$ denote the space of $\mathcal{A}$-linear bounded operators with $\mathcal{A}$-linear bounded adjoints. Then A_0

is a C*-algebra whose group of units $G(A_0)$ retracts onto the subgroup of unitaries $U(A_0) = U(H_{\mathcal{A}})$. In this case we have the space of projections

$$P^*(A_0) = \{p \in A_0 : p = p^2 = p^* \text{ and } p \sim 1 \sim \hat{p}\}, \tag{5.3}$$

where '$\sim$' denotes the Murray–von Neumann equivalence of projections (see e.g. [5]), and we recall that $\hat{p} = 1 - p$.

Compact linear operators on $H_{\mathcal{A}}$ are defined as $\mathcal{A}$-norm limits of finite rank operators (c.f. examples such as the spaces Gr_0 in [35]). The space of such compact operators is denoted by $\mathcal{K}(H_{\mathcal{A}})$. Finite-rank $\mathcal{A}$-submodules of $H_{\mathcal{A}}$ are well-defined and a Fredholm operator on $H_{\mathcal{A}}$ is one whose kernel and image are finite-rank $\mathcal{A}$-submodules. We denote the space of such Fredholm operators by $\mathrm{Fred}(H_{\mathcal{A}})$. There exists a canonical index homomorphism

$$\mathrm{Ind}_{\mathcal{A}} : \mathrm{Fred}(H_{\mathcal{A}}) \longrightarrow K_0(\mathcal{A}), \tag{5.4}$$

where $K_0(\mathcal{A})$ denotes the topological K-group of $\mathcal{A}$ (see e.g. [5]).

5.2 The Grassmannian $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}})$

Since $A_0 = \mathcal{L}(H_{\mathcal{A}})$ is a Banach algebra, we can consider the Grassmannians $\mathrm{Gr}(p, A)$ for suitable $p \in P(A)$. Towards a generalization, we will consider the Banach algebra $\mathcal{L}_J(H_{\mathcal{A}})$ where J is a unitary $\mathcal{A}$-module map, $J^2 = 1$, determining the splitting of Hilbert $\mathcal{A}$-modules $H_{\mathcal{A}} = H_+ \oplus H_-$. An important example is the generalized Fredholm Grassmannian $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}})$ as studied in [21]. It is a Banach manifold modeled on the Banach space $\mathcal{K}(H_{\mathcal{A}})$, and its unitary and topological structures can be determined in a way similar to that of $\widehat{\mathrm{Gr}}(H_+, H)$ (in the case of $\mathcal{A} = \mathbb{C}$) as in [31]. Specifically, we fix a direct sum decomposition of Hilbert $\mathcal{A}$-modules $H_{\mathcal{A}} = H_+ \oplus H_-$, where $H_\pm$ are isomorphic to $H_{\mathcal{A}}$ as $\mathcal{A}$-modules. Then elements of $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}})$ consist of $\mathcal{A}$-submodules W of $H_{\mathcal{A}}$ such that:

(1) The orthogonal projection $\tilde{p}_+ : W \longrightarrow H_+$ is in $\mathrm{Fred}(H_{\mathcal{A}})$, and
(2) The orthogonal projection $\tilde{p}_- : W \longrightarrow H_-$ is in $\mathcal{K}(H_{\mathcal{A}})$.

We define the *big cell* of $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}})$ as the collection of all $\mathcal{A}$-submodules W of $H_{\mathcal{A}}$ such that the projection p_+ is an isomorphism.

Suppose we fix a path component $\widehat{\mathrm{Gr}}(\gamma)$ of $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}})$ corresponding to an element $\gamma \in K_0(\mathcal{A})$ where γ equals the index of p_+ restricted to an element V of this component. Then we can consider pairs $(\alpha, \beta) \in K_0(\mathcal{A}) \times K_0(\mathcal{A})$ for which $\gamma = \alpha - \beta$. Given any such pair, we denote by $B_{\alpha,\beta}$ the subset of all $V \in \widehat{\mathrm{Gr}}(\gamma)$ for which

$$[\ker\, p_+|V] = \alpha, \; [\mathrm{coker}\, p_+|V] = \beta. \tag{5.5}$$

Then following [21] there is a generalized Birkhoff stratification:

$$\widehat{\mathrm{Gr}}(\gamma) = \bigcup_{\alpha,\beta} B_{\alpha,\beta}. \tag{5.6}$$

Furthermore, the $B_{\alpha,\beta}$ are Banach analytic subspaces of $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}})$ and the expression in (5.6) describes a complex analytic stratification of $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}})$.

6 The Conjugation with Generalized Coefficients

6.1 Generalization of Theorem 4.1

Let $\mathcal{A}$ be a (unital) C*-algebra. Given a separable Hilbert space H, we will adopt the Hilbert C*-module of Example 5.1 where we have the unitary equivalence $H \otimes \mathcal{A} \approx H_{\mathcal{A}}$ together with the direct sum decomposition of Hilbert $\mathcal{A}$-modules $H_{\mathcal{A}} = H_+ \oplus H_-$ for which $H_\pm \cong H_{\mathcal{A}}$. In the following we set

$$\widetilde{A} = A \otimes \mathcal{A}, \text{ where } A = \mathcal{L}_J(H). \tag{6.1}$$

Without too much loss of generality, we keep with the separable Hilbert space $H = L^2(S^1, \mathbb{C})$. We will also use the identification $\widehat{\mathrm{Gr}}(H_+, H_{\mathcal{A}}) = \mathrm{Gr}(\tilde{p}, \widetilde{A})$ with $\tilde{p} \in \mathrm{Sim}(\tilde{p}_+, \widetilde{A})$ where $\tilde{p}_+ \in \mathrm{Fred}(H_{\mathcal{A}})$ is the orthogonal projection as previously.

Here we take an operator L of order 1 in the ring $\widehat{\mathbb{A}} \subset \mathbb{B}[\partial^{-1}] \otimes \mathcal{A}$ with coefficients $u_i \in \mathbb{B} \otimes \mathcal{A}$, such that L is given by

$$L = \partial + \sum_{i>-\infty}^{-1} u_i\, \partial^i, \quad (\partial \equiv \partial/\partial x). \tag{6.2}$$

Definition 6.1 Let $\Gamma_+(\mathcal{A})$ be the group of maps $g : D \longrightarrow G(\mathcal{A})$ such that $g(z)$ is holomorphic in $z \in D$, and $g(0) = \mathbf{1}$.

Definition 6.2 Let $\Gamma_-(\mathcal{A})$ be the group of maps $g : \mathbb{C}\backslash\mathrm{Int}\, D \longrightarrow G(\mathcal{A})$ such that $g(z)$ is holomorphic in $z \in \mathbb{C}\backslash\mathrm{Int}\, D$, $g(\infty) = \mathbf{1}$.

Definition 6.3 Let $\Gamma_+^{\widetilde{W}}(\mathcal{A})$ be a dense open subset of $\Gamma_+(\mathcal{A})$ defined by

$$\Gamma_+^{\widetilde{W}}(\mathcal{A}) = \{g \in \Gamma_+(\mathcal{A}) : g^{-1}\widetilde{W} \text{ is transverse to } H_-\}.$$

Lemma 6.1 The group $\Gamma_-(\mathcal{A})$ acts freely and transitively on $\mathrm{Gr}(\tilde{p}, \widetilde{A})$.

Proof Firstly, by representing $\mathcal{A}$ as a C*-subalgebra of $\mathcal{L}(E)$ for some Hilbert space E, we can realize $H_{\mathcal{A}} \cong L^2(S^1, \mathcal{A})$ in order to define the concept of a measurable $\mathcal{A}$-valued function. Thus $\mathcal{A}$ becomes a C*-subalgebra of $\mathcal{L}(L^2(S^1, E))$ by acting diagonally. In this way we see that $C(S^1, \mathcal{A})$ acts on $H_{\mathcal{A}}$, and hence so do $\Gamma_\pm(\mathcal{A})$. To see that $\Gamma_-(\mathcal{A})$ acts transitively, observe that removal of a finite number of z^ν for $\nu \geq 0$, and replacing these by a finite number of z^ν for $\nu < 0$, we eventually obtain subspaces $\widetilde{W}'$ in the orbit of $\Gamma_-(\mathcal{A})$ which are dense subspaces in $\mathrm{Gr}(\tilde{p}, \widetilde{A})$. So $\Gamma_-(\mathcal{A})$ acts transitively, essentially by adding ever more powers of z . Observe that $H_{\mathcal{A}}$ is a right $\mathcal{A}$-submodule and $\Gamma_-(\mathcal{A})$ acts on the left in producing a right $\mathcal{A}$-submodule.

Given that $\Gamma_-(\mathcal{A})$ acts transitively, if $g \in \Gamma_-(\mathcal{A})$, and $\widetilde{W} \in \mathrm{Gr}(\tilde{p}, \widetilde{A})$ with $g\widetilde{W} = \widetilde{W}$, then let us choose $h \in \Gamma_-(\mathcal{A})$ such that $hH_+ = \widetilde{W}$. We then have $(h^{-1}gh)H_+ = H_+$, and since $\mathbf{1} \in H_+$, it follows that $h^{-1}gh \in H_+$. Then, all the negative Fourier coefficients for $h^{-1}gh$ must vanish, and therefore $h^{-1}gh = 1$. This immediately yields $g = 1$, and therefore the action of $\Gamma_-(\mathcal{A})$ is free as well as transitive. □

We next define a Baker function in relationship to this generalization

Definition 6.4 Taking $\widetilde{W} \in \mathrm{Gr}(\tilde{p}, \widetilde{A})$ in the big cell and $g \in \Gamma_+(\boldsymbol{\mathcal{A}})$, consider the orthogonal projection

$$\tilde{p}_+^g \ : \ g^{-1}(\widetilde{W}) \longrightarrow H_+ \ , \ \tilde{p}_+^g \in \mathrm{Fred}(H_{\mathcal{A}}). \tag{6.3}$$

The Baker function associated to $\widetilde{W}$ is defined by

$$\psi_{\widetilde{W}} = (\tilde{p}_+^g)^{-1}(\mathbf{1} \otimes \boldsymbol{\mathcal{A}}) = \widetilde{W} e^{xz}. \tag{6.4}$$

Explicitly,

$$\psi_{\widetilde{W}} = \psi_{\widetilde{W}}(x, z) = e^{xz} \left(1 + \sum_{i=1}^{\infty} a_i(x)\, z^{-i} \right), \tag{6.5}$$

where now the a_i are analytic functions on $\Gamma_+^{\widetilde{W}}(\boldsymbol{\mathcal{A}})$ extending to meromorphic functions on all of $\Gamma_+(\boldsymbol{\mathcal{A}})$.

Theorem 6.1 *Given the Baker function $\psi_{\widetilde{W}}$ associated to a subspace $\widetilde{W} \in \mathrm{Gr}(\tilde{p}, \widetilde{A})$, the ring $\widehat{\mathbb{A}} \subset \mathbb{B}[\partial^{-1}] \otimes \boldsymbol{\mathcal{A}}$, conjugates into the Banach algebra $\widetilde{A}$ as a subring up to constant coefficient operators.*

Proof We follow the principles used in establishing Theorem 4.1, except that the coefficient variables are taken to be in $\boldsymbol{\mathcal{A}}$, and we work with the $\Gamma_{\pm}(\boldsymbol{\mathcal{A}})$ and $\Gamma_+^{\widetilde{W}}(\boldsymbol{\mathcal{A}})$. To proceed, we consider $K \in \mathbb{B}[\partial^{-1}] \otimes \boldsymbol{\mathcal{A}}$ as given by

$$K = 1 + \sum_{i=1}^{\infty} a_i(x)\, \partial^{-i}. \tag{6.6}$$

Essentially the same argument used in proving Theorem 4.1 implies the conjugation property, namely $L = K(\partial)K^{-1}$.

Since $\widetilde{W} \in \mathrm{Gr}(\tilde{p}, \widetilde{A})$ belongs to the big cell, the projection $\tilde{p}_+ \in \mathrm{Fred}(H_{\mathcal{A}})$, is an isomorphism. Accordingly, we have $\widetilde{W} = gh_g H_+$ with $h_g \in \Gamma_-(\boldsymbol{\mathcal{A}})$. We recall now the Baker function $\psi_{\widetilde{W}}$ of Definition 6.4. Since by Lemma 6.1 the group $\Gamma_-(\boldsymbol{\mathcal{A}})$ acts freely and transitively on $\mathrm{Gr}(\tilde{p}, \tilde{A})$, and as L involves only derivatives with respect to the x-variable, we obtain as before $L\psi_{\widetilde{W}} = z\psi_{\widetilde{W}}$. The remainder of the proof now follows by applying the relevant parts of the proof of Theorem 4.1. with the obvious modifications. □

Remark 6.1 Recalling the earlier discussion, observe that the projection $\tilde{p}_+^g \in \mathrm{Fred}(H_{\mathcal{A}})$ has a well-defined index $\alpha = \mathrm{Ind}\ \tilde{p}_+^g \in K_0(\boldsymbol{\mathcal{A}})$, and accordingly the function $\psi_{\widetilde{W}} = (\tilde{p}_+^g)^{-1}(\mathbf{1} \otimes \boldsymbol{\mathcal{A}})$ is indexed by $\alpha \in K_0(\boldsymbol{\mathcal{A}})$.

7 Semigroups and a Parametrized Plücker Embedding

7.1 The Plücker Embedding

We begin by reviewing the case $\boldsymbol{\mathcal{A}} = \mathbb{C}$. The (antisymmetric) Fock space of H (see e.g. [31]) is defined by

$$\mathbb{F}(H) = \bigoplus_{n \geq 0} \Lambda^n(H), \qquad \Lambda^0(H) = \mathbb{C},\ \Lambda^1(H) = H. \tag{7.1}$$

Let $\Lambda_\pm = \mathbb{F}(H_\pm)$. Recalling $A = \mathcal{L}_J(H)$, a 'determinant' mapping, $\det : V(p, A) \to \Lambda_+$, is defined as follows. Here an element $w \in V(p, A)$ is taken to be an admissible basis as above, so that given the pair $(w, \lambda) \in V(p, A) \times \mathbb{C}$, its image in Λ_+ is expressed as

$$\lambda \sum_{[i]\in\mathcal{S}} (\det w_{[i]})\ f_{j_1} \wedge f_{j_2} \wedge \cdots\ , \tag{7.2}$$

where $\mathcal{S}$ is a certain indexing set [31]. Since (7.2) is defined up to a (nonzero) determinant, the image is a line in Λ_+ and so we have produced a well-defined map

$$\det : V(p, A) \to \mathbb{P}(\Lambda_+) \cong \mathbb{P}(\ell^2(\mathcal{S})), \tag{7.3}$$

which sends an admissible basis to a point in projective Fock space $\mathbb{P}(\Lambda_+)$. In particular, (4.18) defines a global admissible basis (see [2, 32, 35]) and det in (7.2) is definable via the Wronskian

$$(\psi_W \wedge \partial\psi_W \wedge \cdots \wedge \partial^{k-1}\psi_W \wedge \cdots)|_{x=0}. \tag{7.4}$$

On setting $B = \mathcal{L}_{\mathrm{d}(J)}(\Lambda_+)$ where $\mathrm{d}(J)$ denotes the induced operator on Λ_+ under det of the unitary J, we identify $\mathbb{P}(\Lambda_+)$ with $\mathrm{Gr}(q, B)$, and thus we have the homogeneous space representation

$$\mathbb{P}(\Lambda_+) = \mathrm{Gr}(q, B) = V(q, B)/G(q), \tag{7.5}$$

where q denotes the (unique) rank 1 projection in B corresponding to a line ℓ in Λ_+ . For an admissible basis $w \in V(p, A)$, the composition of assignments $w \mapsto \det w \mapsto q$, defines an analytic injective map

$$V(p, h) : V(p, A) \to P(B) \subset B, \tag{7.6}$$

which descends to analytic map

$$\mathrm{Gr}(p, h) : \mathrm{Gr}(p, A) \longrightarrow \mathrm{Gr}(q, B). \tag{7.7}$$

The map $\mathrm{Gr}(p, h)$ is actually a holomorphic embedding of complex Hilbert manifolds and is usually referred to as *the infinite dimensional Plücker embedding* (see [25, 31, 35]). Relative to (7.7) we will need the following technical feature. Let $W = p(H) \in \mathrm{Gr}(p, A)$ and $\ell = q(B) \in \mathrm{Gr}(q, B)$. Since $\mathrm{Gr}(p, h)$ is a holomorphic embedding, it follows from (3.7) that the map of complex derivatives

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}}(W, W^\perp) &\longrightarrow \mathrm{Hom}_{\mathbb{C}}(\ell, \ell^\perp), \\ pA\hat{p} &\longrightarrow qB\hat{q}, \end{aligned} \tag{7.8}$$

is injective. Equivalently, the embedding is induced by the (holomorphic) sections of the corresponding holomorphic line bundle $\mathrm{DET}^* \longrightarrow \mathrm{Gr}(p, A)$ endowed with a Hilbert space structure on sections.

7.2 The τ-function

In the analysis and transformations of integrable systems the Plücker embedding equations explicitly describe the orbit of the vacuum state (see e.g. [20, 27]). From the embedding equations an essential feature known as the τ-*function* is derived. We recall now its definition. Relative to an admissible basis represented by $w \in V(p, A)$, the embedding $\mathrm{Gr}(p, h)$ leads to a function

$$\mathrm{Gr}_w(p, h)(w') = \det \langle w, w' \rangle, \tag{7.9}$$

where $\langle w, w' \rangle$ denotes the matrix whose (i, j)-entry is $\langle w_i, w'_j \rangle$. Denoting the vacuum state (the canonical section of DET^*) by $\mathrm{Gr}_{\mathbf{1}}(p, h)$, the τ*-function* is defined as

$$\tau_w(g) := \langle \mathrm{Gr}_{\mathbf{1}}(p, h),\ g\,\mathrm{Gr}_w(p, h) \rangle, \qquad g \in \Gamma_+. \tag{7.10}$$

We recall that the Baker function ψ_W has the property of extending to an analytic function of z in $|z| > 1$ (fixing $g \in \Gamma_+^W$). For $\zeta \in \mathbb{C}$ satisfying $|\zeta| > 1$, the map defined by $q_\zeta(z) = 1 - z\zeta^{-1}$, leads to the notable relationship between the Baker and τ-functions [35] Proposition 5.14:

$$\psi_W(g, \zeta) = \tau_W(g q_\zeta)(\tau_W(g))^{-1}. \tag{7.11}$$

7.3 Semigroups and the Determinant

To proceed, let us consider an ambient subsemigroup structure about $V(p, A)$ along with its induced image under det. Specifically, we introduce multiplicative subsemigroups denoted S and T such that $V(p, A) \subset S \subseteq A$, where S consists of elements of A admitting a determinant (recall that $V(p, A)$ admits determinants via the admissible bases), and T is the image of S under det, so that $\det(V(p, A)) \subset T \subseteq B$. This provides an analytic multiplicative subsemigroup homomorphism $h : S \longrightarrow T$ induced by det, where for $p \in \mathrm{Sim}(p^+, A)$, $q = h(p)$, we have a commutative diagram with vertical maps inclusions

$$\begin{array}{ccc} S & \xrightarrow{\quad h \quad} & T \\ \uparrow & & \uparrow \\ V(p, A) & \xrightarrow{V(p,h)=\det} & V(q, B). \end{array} \tag{7.12}$$

Specifically, the homomorphism h can be identified with the infinite exterior product Λ^∞ leading to a continuous homomorphism of multiplicative semigroups. Note that $q = h(p)$ is identifiable with the rank 1 projection in $P(\mathcal{L}(\Lambda_+))$ induced by det as previously.

We recall $\widetilde{A} = A \otimes \boldsymbol{\mathcal{A}}$ where $\boldsymbol{\mathcal{A}}$ is a (unital) C*-algebra, and set $\widetilde{B} = B \otimes \boldsymbol{\mathcal{A}}$.

Given $w \in V(p, A) \subset A$ and $a \in \boldsymbol{\mathcal{A}}$, consider the assignment $w \otimes a \mapsto (\det w) \otimes a$ extending det as $\boldsymbol{\mathcal{A}}$-valued. Further, we set $\widetilde{S} = S \otimes \boldsymbol{\mathcal{A}}$ and $\widetilde{T} = T \otimes \boldsymbol{\mathcal{A}}$, and let $\tilde{h} : \widetilde{S} \longrightarrow \widetilde{T}$ be the multiplicative subsemigroup homomorphism induced by the infinite exterior product Λ^∞. By the essential functoriality of Λ^∞, the following diagram

$$\begin{array}{ccc} S & \xrightarrow{h} & T \\ \downarrow & & \downarrow \\ \widetilde{S} & \xrightarrow{\tilde{h}} & \widetilde{T} \end{array} \tag{7.13}$$

commutes where the vertical arrows represent inclusions into their respective tensor products.

Continuing, let

$$\widetilde{\det} = \tilde{h}|V(\tilde{p}, \widetilde{A}) : V(\tilde{p}, \widetilde{A}) \longrightarrow V(\tilde{q}, \widetilde{B}), \tag{7.14}$$

where relative to the orthogonal projection $\tilde{p}_+ \in \mathrm{Fred}(H_{\mathcal{A}})$, we have taken $\tilde{p} \in \mathrm{Sim}(\tilde{p}_+, \widetilde{A})$, and $\tilde{q} = \tilde{h}(\tilde{p})$. Using the same principles as before we obtain a commutative diagram

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\tilde{h}} & \widetilde{T} \\ \uparrow & & \uparrow \\ V(\tilde{p}, \widetilde{A}) & \xrightarrow{V(\tilde{p},\tilde{h})=\widetilde{\det}} & V(\tilde{q}, \widetilde{B}) \end{array} \tag{7.15}$$

Lemma 7.1 Let $h : \widetilde{S} \longrightarrow \widetilde{T}$ be the multiplicative semigroup homomorphism which, as above, assigns a given $\tilde{p} \in P(\widetilde{S})$ to a rank 1 projection $\tilde{q} = \tilde{h}(\tilde{p})$ in $P(\widetilde{T})$. Then the following diagram is commutative:

$$\begin{array}{ccc} V(\tilde{p}, \widetilde{A}) & \xrightarrow{V(\tilde{p},\tilde{h})} & V(\tilde{q}, \widetilde{B}) \\ \Pi \downarrow & & \downarrow \Pi \\ \mathrm{Gr}(\tilde{p}, \widetilde{A}) & \xrightarrow{\mathrm{Gr}(\tilde{p},\tilde{h})} & \mathrm{Gr}(\tilde{q}, \widetilde{B}) \end{array} \tag{7.16}$$

In particular, if $\tilde{h}$ is analytic (resp. smooth), then the maps $V(\tilde{p}, \tilde{h})$ and $\mathrm{Gr}(\tilde{p}, \tilde{h})$ are also analytic (resp. smooth).

Proof Keeping in mind the commutativity of (7.15), we can take $\tilde{h}|V(\tilde{p}, \widetilde{A})$ and restrict matters to $\widetilde{\det}$. Then it follows from [10] Theorem 7.1 that if $\tilde{p}, \tilde{p}_1 \in P(\widetilde{A})$ and $\Pi(\tilde{p}_1) \in \mathrm{Gr}(\tilde{p}, \widetilde{A})$, the restriction $\Pi|P(\widetilde{A}) \cap V(\tilde{p}_1, \widetilde{A})$ is an analytic diffeomorphism onto an open subset of $\mathrm{Gr}(\tilde{p}, \widetilde{A})$ containing $\Pi(\tilde{p}_1)$, thus providing a natural analytic local section for the analytic map $\Pi : P(\widetilde{A}) \longrightarrow \mathrm{Gr}(\widetilde{A})$ through $\tilde{p}_1 \in P(\widetilde{A})$.

Since the spaces $V(\tilde{p}, \widetilde{A})$, $\mathrm{Gr}(\tilde{p}, \widetilde{A})$, etc. are defined by the multiplicative structure, then the above diagram commutes for such a multiplicative map as given. Thus when $\tilde{h}$ is analytic, $V(\tilde{p}, \tilde{h})$, and consequently $\mathrm{Gr}(\tilde{p}, \tilde{h})$, are also analytic; the same argument applies when $\tilde{h}$ is smooth. □

7.4 A Parametrized Plücker Embedding

Suppose now $\boldsymbol{\mathcal{A}}$ is a (unital) commutative C*-algebra. Then $\boldsymbol{\mathcal{A}} \cong C(Y)$, where Y is a compact Hausdorff space. In which case (6.1) shapes up as

$$\begin{aligned}\widetilde{A} = A \otimes \boldsymbol{\mathcal{A}} &= \{\text{continuous functions } Y \longrightarrow A = \mathcal{L}_J(H)\}\\ &\cong \mathcal{L}_J(H_{\boldsymbol{\mathcal{A}}}) \cap \{\text{continuous functions } Y \longrightarrow \mathcal{L}(H)\},\end{aligned} \tag{7.17}$$

for which the $\| \ \|_2$-trace in (3.4) is regarded as continuous as a function of Y. We make the appropriate parametrization by Y of the groups $\Gamma_{\pm}$ in Section 4 by specializing Definitions 6.1–6.3 accordingly:

Definition 7.1 Let $\Gamma_+(Y)$ be the group of maps $g : D \times Y \longrightarrow \mathbb{C}^*$ such that g is continuous in y for each $y \in Y$, $g(z, y)$ is holomorphic in $z \in D$, and $g(0, y) = 1$, for each $y \in Y$.

Definition 7.2 Let $\Gamma_-(Y)$ be the group of maps $g : (\mathbb{C} \backslash \mathrm{Int}\ D) \times Y \longrightarrow \mathbb{C}^*$ such that g is continuous in y for each $y \in Y$, $g(z, y)$ is holomorphic in $z \in \mathbb{C} \backslash \mathrm{Int}\ D$, and $g(\infty, y) = 1$, for each $y \in Y$.

Definition 7.3 Let $\Gamma_+^{\widetilde{W}}(Y)$ be a dense open subset of $\Gamma_+(Y)$ defined by

$$\Gamma_+^{\widetilde{W}}(Y) = \{g \in \Gamma_+(Y) : g^{-1}\widetilde{W} \text{ is transverse to } H_-\}.$$

Theorem 7.1 *The map* $\widetilde{\det} = V(\tilde{p}, \tilde{h}) : V(\tilde{p}, \widetilde{A}) \longrightarrow V(\tilde{q}, \widetilde{B})$ *induces an analytic (i.e. holomorphic) embedding*

$$\mathrm{Gr}(\tilde{p}, \tilde{h}) : \mathrm{Gr}(\tilde{p}, \widetilde{A}) \longrightarrow \mathrm{Gr}(\tilde{q}, \widetilde{B}). \tag{7.18}$$

In fact, the map $\mathrm{Gr}(\tilde{p}, \tilde{h})$ *is realized as a family of* Y*-parametrized Plücker embeddings given by the map* $\mathrm{Gr}(p, h)$ *in* (7.7).

Proof Taking $\widetilde{W} \in \mathrm{Gr}(\tilde{p}, \widetilde{A})$ in the big cell, let us consider the Baker function $\psi_{\widetilde{W}}$ given by

$$\psi_{\widetilde{W}} = \psi_{\widetilde{W}}(x, y, z) = e^{xz}\left(1 + \sum_{i=1}^{\infty} a_i(x, y)\, z^{-i}\right), \qquad (x, y) \in \mathbb{C} \times Y, \tag{7.19}$$

where now the a_i are analytic functions on $\Gamma_+^{\widetilde{W}}(Y)$ extending to meromorphic functions on all of $\Gamma_+(Y)$. By Lemma 6.1, $\Gamma_-(Y)$ acts freely and transitively on $\mathrm{Gr}(\tilde{p}, \widetilde{A})$, and so convergence and differentiation is then uniform over the Y-variables, essentially by extending the action of Γ_- pointwise. Since we are dealing only with derivatives at $x = 0$, and this differentiation is uniform over the (parameter) Y-variables, (7.19) (c.f. (4.18)) then yields a global admissible basis for $\widetilde{W}$ as given by the vectors

$$(\psi_{\widetilde{W}}, \partial\psi_{\widetilde{W}}, \ldots, \partial^{k-1}\psi_{\widetilde{W}}, \ldots)|_{x=0}. \tag{7.20}$$

Then $\widetilde{\det} : V(\tilde{p}, \widetilde{A}) \longrightarrow V(\tilde{q}, \widetilde{B})$ is defined via the Wronskian

$$(\psi_{\widetilde{W}} \wedge \partial\psi_{\widetilde{W}} \wedge \cdots \wedge \partial^{k-1}\psi_{\widetilde{W}} \wedge \cdots)|_{x=0}. \tag{7.21}$$

Now we already know that $h : S \longrightarrow T$ is injective, and by the property of Λ^∞, then so too is $\tilde{h} : \widetilde{S} \longrightarrow \widetilde{T}$. In particular, the analyticity of h induces the same for $\tilde{h}$. Thus by Lemma 7.1, $\mathrm{Gr}(\tilde{p}, \tilde{h})$ is analytic and injective. With respect to complex derivatives we already have $pA\hat{p} \longrightarrow qB\hat{q}$ injective, and by extension to the tensor product, the same is true for $\tilde{p}\widetilde{A}(\widehat{\tilde{p}}) \longrightarrow \tilde{q}\widetilde{B}(\widehat{\tilde{q}})$ which proves that $\mathrm{Gr}(\tilde{p}, \tilde{h})$ is an analytic embedding. Since by construction $y \in Y$ is effectively a parameter throughout, the last statement follows. □

Remark 7.1 Theorem 7.1 can be compared with the results of [12] Section 4 which give some criteria for embeddings of Grassmannians over arbitrary Banachable algebras.

Remark 7.2 Note that in view of Theorem 7.1, the prevailing relation (7.11) between the Baker and τ-function extends to a Y-parametrized relation.

8 Towards an Operator-valued Baker Function

In this section we outline an abstraction at the level of operator-valued functions. An astute-minded reader will see that the idea requires replacing the variable z by a single operator.

8.1 Laurent Series Generator

Definition 8.1 [4] Let A be a unital Fréchet algebra. An invertible element $\zeta \in G(A) \subset A$ is said to be a *Laurent series generator for* A if each $a \in A$ is expressible as

$$a = \sum_{i=-\infty}^{\infty} a_i\, \zeta^i, \tag{8.1}$$

for scalars a_i, and the series converges absolutely with respect to each continuous seminorm on A. We say that A has the *unique expression property* if such a representation is always unique.

As shown in [4], such algebras A with a Laurent series generator include algebras which are isomorphic to various types of function algebras defined on S^1 or on the annulus.

8.2 Fredholm Operators on Banach Spaces

The next step involves introducing from [37] the notion of compact and Fredholm operators between complex Banach spaces E and E'. Let $\mathcal{K}(E, E')$ denote the compact operators. The more general meaning of a Fredholm operator is here based on the notion of 'right and left aggregation'; we refer to [37] for details. An operator $T \in \text{Fred}(E)$ is stable under compact perturbations and admits a well-defined *index* given by $\text{Ind}(T) = \dim \text{Ker}\ T - \text{codim}\ \text{Im}\ T$. The index $\text{Ind}(T)$ is constant on connected components, is invariant under compact perturbations, and satisfies $\text{Ind}(T_1 T_2) = \text{Ind}(T_1) + \text{Ind}(T_2)$. Moreover, there is an induced homomorphism $\text{Ind} : \text{Fred}(E) \to \mathbb{Z}$.

Let E be a complex Banach space admitting a decomposition of the type (2.10). Let

$$\widehat{G} \subset \left\{ \begin{bmatrix} T_1 & *_1 \\ *_2 & T_2 \end{bmatrix} : T_1 \in \text{Fred}(F),\ T_2 \in \text{Fred}(F^c) \right\}, \tag{8.2}$$

be a Banach Lie group that generates a Banach algebra A acting on E, but with possibly a different norm. Here, $*_i$ for $i = 1, 2$ tentatively represent operators on E of some specified class.

Suppose that $\widehat{G}$ acts analytically on $\text{Gr}(A)$ with a typical orbit denoted by $\widehat{\text{Gr}}(A)$. Fixing $p \in P(A)$, let $\widehat{\text{Gr}}(p, A) = \widehat{\text{Gr}}(A) \cap \text{Gr}(p, A)$, and let $\widehat{\text{Gr}}_\alpha(A)$ denote a connected component of $\widehat{\text{Gr}}(A)$ for which $\text{Ind}\ T_1 = \alpha$. Accordingly, we define $\widehat{\text{Gr}}_\alpha(p, A) = \widehat{\text{Gr}}_\alpha(A) \cap \text{Gr}(p, A)$. The restriction $V_\alpha(p, A) = V(p, A)|\ \widehat{\text{Gr}}_\alpha(p, A)$, thus provides a framing for elements of $\widehat{\text{Gr}}_\alpha(p, A)$ [11].

8.3 Conjugation into the Banach Algebra A

Suppose now that A is a unital Banach algebra with Laurent series generator ζ (with the unique expression property) acting cyclically on E. We observe that the idempotents (projections) of A under the action form certain subspaces of the complex Banach space E. We shall assume a decomposition as in (2.10), $E = F \oplus F^c$ where the closed (splitting) subspaces F and F^c are specified as follows.

Let ϕ be a cyclic vector for this action and take F to be the closed linear span of all vectors $\zeta^i \phi$ for $i \geq 0$, and F^c the closed linear span of $\zeta^i \phi$ for $i < 0$. With regards to (8.2), we now take the operators $*_1 \in \mathcal{K}(F^c, F)$ and $*_2 \in \mathcal{K}(F, F^c)$.

Here we assume that a fixed $p \in P(A)$ acts as the projection of E on F along F^c. We take $\widehat{\text{Gr}}(p, A)$ to be the Grassmannian consisting of subspaces $W \in \text{Gr}(F, E)$ where $W = q(E)$ for $q \in P(A)$ such that:

(1) The projection $p_1 = pq\ :\ W \longrightarrow F$ is in $\text{Fred}(E)$, and
(2) The projection $p_2 = (1 - p)q\ :\ W \longrightarrow F^c$ is in $\mathcal{K}(E)$.

We define the *big cell* of $\widehat{\text{Gr}}(p, A)$ as the collection of all subspaces W of E such that p_1 is an isomorphism.

Let D be the closed unit disc centered at the origin in $\mathbb{C}$ which contains the spectrum of the generator ζ. We define $\widetilde{\Gamma}_+$ to be the group of (invertible) holomorphic maps $g : D \longrightarrow G(A)$, such that $g(0) = 1$, and define an action of $\widetilde{\Gamma}_+$ on F by $g \cdot v = g(\zeta)v$ where $g(\zeta)$ is given by the holomorphic functional calculus. We also define $\widetilde{\Gamma}_-$ as the group of holomorphic maps of the form $g(\frac{1}{z})$ where $g \in \widetilde{\Gamma}_+$, and it is assumed that $\widetilde{\Gamma}_-$ acts freely and transitively on $\widehat{\mathrm{Gr}}(F, E) = \widehat{\mathrm{Gr}}(p, A)$, where $p \in \mathrm{Sim}(p_1, A)$.

As previously, we consider subspaces $W \in \widehat{\mathrm{Gr}}(p, A)$ of the form

$$W = g h_g F, \tag{8.3}$$

with $g \in \widetilde{\Gamma}_+$ and $h_g \in \widetilde{\Gamma}_-$. Also for $g \in \widetilde{\Gamma}_+$, we consider projections

$$p_1^g \; : \; g^{-1}(W) \longrightarrow F \; , \; \; p_1^g \in \mathrm{Fred}(E), \tag{8.4}$$

and define

$$\widetilde{\Gamma}_+^W = \{g \in \widetilde{\Gamma}_+ : p_1^g \text{ is an isomorphism}\}. \tag{8.5}$$

The operator-valued Baker function associated to W is defined as:

$$\begin{aligned} \psi_W &= (p_1^g)^{-1}(\mathbf{1}) = W e^{x\zeta}, \\ &= g(z)(1 + \sum_{i=1}^{\infty} a_i(g)\zeta^{-i}), \end{aligned} \tag{8.6}$$

where $g \in \widetilde{\Gamma}_+^W$ and the a_i are analytic functions on $\widetilde{\Gamma}_+^W$ extending to meromorphic functions on all of $\widetilde{\Gamma}_+$. By the same principles of Section 4 we obtain the equation $L\psi_W = \zeta\psi_W$. Let $\alpha = \mathrm{Ind}\ p_+^g$ and recall $\widehat{\mathbb{A}} \subset \mathbb{B}[\partial^{-1}]$. Adopting the same strategies used in establishing Theorems 4.1 and 6.1, leads to:

Proposition 8.1 Let A be a unital Banach algebra with Laurent series generator ζ (with the unique expression property) acting cyclically on E. Given the Baker function ψ_W associated to a subspace $W \in \widehat{\mathrm{Gr}}_\alpha(p, A)$ as given by (8.3), the assignment $\partial^{-1} \mapsto \zeta$ induces a conjugation of $\widehat{\mathbb{A}}$ into a subring of A up to constant coefficient operators.

9 Rings of Formal Pseudodifferential Operators in Several Variables

9.1 Iterated Laurent Series

The several variables case differs implicitly from the 1-variable case considered so far, and requires a different Grassmann model. We regard this as an area of potential application of our extended coefficients. For now, we encode an existing algebraic construction into our framework that we expect to develop in a future work. Our approach is based to an extent on [28, 29] and also in view of related work of other authors that is surveyed in [33] to which we refer for details.

Let $\mathcal{A}_0$ be an associative ring with derivation $\mathrm{d} : \mathcal{A}_0 \longrightarrow \mathcal{A}_0$. As a generalization of an earlier section, we consider the ring $\mathcal{A}_0[\partial^{-1}]$ of formal pseudodifferential operators in $\mathcal{A}_0$ as a left $\mathcal{A}_0$-module of all formal expressions:

$$L = \sum_{i>-\infty}^{N} a_i\, \partial^i\ ,\ a_i \in \mathcal{A}_0. \tag{9.1}$$

For a given $a \in \mathcal{A}_0$, we have

$$\begin{aligned} [\partial, a] &= \partial a - a\partial = \mathrm{d}(a), \\ [\partial^{-1}, a] &= \partial^{-1} a - a\partial^{-1} = -\mathrm{d}(a)\partial^{-2} + \mathrm{d}^2(a)\partial^{-3} - \dots, \end{aligned} \tag{9.2}$$

Following [28], $\mathcal{A}_0[\partial^{-1}]$ is an associative ring and moreover, the above construction can be iterated. Given N variables $x_1, \dots x_N$, let $\mathbb{C}((x_1)) \dots ((x_N))$ be the N-dimensional local field of iterated Laurent series where at stage i, the space $\mathbb{C}((x_1)) \dots ((x_i))$ is defined to be the quotient field of $\mathbb{C}((x_1)) \dots ((x_{i-1}))[[x_i]]$.

On setting $\partial_i = \partial/\partial x_i$, for $1 \leq i \leq N$, we let

$$\begin{aligned} \mathcal{Q} &= \mathbb{C}((x_1)) \dots ((x_N))((\partial_1^{-1})) \dots ((\partial_N^{-1})), \\ \mathcal{E} &= \mathbb{C}[[x_1, \dots, x_N]]((\partial_1^{-1})) \dots ((\partial_N^{-1})). \end{aligned} \tag{9.3}$$

Then $\mathcal{E} \subset \mathcal{Q}$ is a subring and there is the following decomposition into non-negative $(+)$ and negative $(-)$ orders of operators as given by:

$$\mathcal{Q} = \mathcal{Q}_+ + \mathcal{Q}_-\ ,\ \mathcal{E} = \mathcal{E}_+ + \mathcal{E}_-,\ \mathcal{E}_\pm = \mathcal{E} \cap \mathcal{Q}_\pm. \tag{9.4}$$

Following the conjugacy theorem of [28] (Theorem 1), we have:

Theorem 9.1 [28] *Let $L_1 \in \partial_1 + \mathcal{E}_-, \dots,\ L_N \in \partial_N + \mathcal{E}_-$ be operators satisfying the commutativity relations $[L_i, L_j] = 0$, for $1 \leq i, j \leq N$. Then there exists an operator $K \in 1 + \mathcal{E}_-$, such that*

$$L_1 = K(\partial_1)K^{-1}\ , \dots,\ L_N = K(\partial_N)K^{-1}\ . \tag{9.5}$$

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index such that $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}$. We will proceed to the $\mathbb{C}[[x]]$-module of partial differential operators

$$\mathcal{P} = \left\{ \sum_\alpha a_\alpha\, \partial^\alpha\ :\ a_\alpha \in \mathcal{A}_0\ ,\ \alpha \in \mathbb{Z} \right\} \subseteq \mathcal{E}. \tag{9.6}$$

Since $[\partial_i, \partial_j] = 0$, it is a consequence of the Leibnitz rule that $\mathcal{P}$ is endowed with a $\mathbb{C}$-algebra structure. The *residue* of the operator $L \in \mathcal{P}$ is defined as

$$\mathrm{res}_{\mathcal{P}}(L) = a_{-1\ \dots\ -1} \in \mathbb{C}, \tag{9.7}$$

and for $L, M \in \mathcal{P}$, the pairing

$$\langle L, M \rangle_{\mathcal{P}} = \mathrm{res}_{\mathcal{P}}(LM), \tag{9.8}$$

defines a nondegenerate bilinear form on $\mathcal{P}$ [28] (c.f. [34]).

In the following, $\mathbb{C}[[t]]$ denotes $\mathbb{C}[[\{t_\alpha\}]]$ the ring of Taylor series in $\{t_\alpha\}$, $\partial_\alpha = \frac{\partial}{\partial t_\alpha}$, and we take L^α to denote $L_1^{\alpha_1} \cdots L_N^{\alpha_N}$.

The KP-hierarchy for N variables consists of the following Lax system

$$\partial_\alpha L_i = [(L^\alpha)_+\, ,\ L_i],$$
$$[L_i, L_j] = 0 \text{ (commutativity relations)}, \tag{9.9}$$

for $1 \leq i, j \leq N$, $0 \subset \alpha$, and for which

$$L_i = \partial_i + \sum_\alpha u_{i\alpha}(t)\ \partial^\alpha \in \mathcal{P} \otimes \mathbb{C}[[t]]. \tag{9.10}$$

Let V denote the vector space $\mathbb{C}((x_1))\ldots((x_N))$ of the iterated Laurent series, with its filtration $V_n = x_N^n\, \mathbb{C}((x_1))\ldots((x_{N-1}))[[x_N]]$, for $n \in \mathbb{Z}$.

Suppose now $\boldsymbol{\mathcal{A}}$ is a (unital) commutative $*$-algebra. Then $A = V \otimes \boldsymbol{\mathcal{A}}$ becomes an $\boldsymbol{\mathcal{A}}$-module, and via the spatial correspondence, we have the Grassmannian $\mathrm{Gr}(\mathcal{L}(V \otimes \boldsymbol{\mathcal{A}})) \cong \mathrm{Gr}(A)$.

Now we observe the following data (**) relative to the systems (9.9) and (9.10):

1. Let $W_0 = \mathbb{C}[x_1^{-1}, \ldots, x_N^{-1}] \otimes \boldsymbol{\mathcal{A}}$, and let $W_0^c \subset V \otimes \boldsymbol{\mathcal{A}}$ be a complementary subspace such that $W_0 \oplus W_0^c \cong V \otimes \boldsymbol{\mathcal{A}}$.
2. Let $F_0 = \mathrm{Hom}(W_0, \widehat{W}_0^c) \subset \mathrm{Gr}(V \otimes \boldsymbol{\mathcal{A}})$, where $\widehat{W}_0^c$ is a commensurable subspace formed by taking a certain inverse limit just as in [29] (note that F_0 here plays the role of the 'big cell').
3. We have a nondegenerate bilinear $\boldsymbol{\mathcal{A}}$-valued form on $\mathcal{P} \otimes \boldsymbol{\mathcal{A}}$ defined by:

$$\langle L \otimes a\, ,\ M \otimes b \rangle\ =\ \langle L, M \rangle_{\mathcal{P}}\ (a^* b)\ ,$$

for $L, M \in \mathcal{P}$ and $a, b\ \in \boldsymbol{\mathcal{A}}$, that induces the same $\langle\ ,\ \rangle$ on $V \otimes \boldsymbol{\mathcal{A}}$, via $\partial^\alpha \mapsto x_\alpha$.

Theorem 9.2 *Let $\boldsymbol{\mathcal{A}}$ be a (unital) commutative $*$-algebra. In relationship to the Lax systems* (9.9) *and* (9.10)*, we recall the data (**) as above. Then there exists for a given $W \in F_0 \subset \mathrm{Gr}(A)$, a Baker (eigen)function ψ_W which induces a conjugation of the ring $\mathcal{P} \otimes \mathbb{C}[[t]] \otimes \boldsymbol{\mathcal{A}}$ of the operators L_i, as a subring of constant coefficient operators within the algebra $A = \mathcal{L}(V \otimes \boldsymbol{\mathcal{A}})$.*

Proof The proof is a straightforward modification of the techniques of [29]. Firstly, we establish a preliminary conjugation result. Consider the operators

$$L_i = \partial_i + \sum_\alpha u_{i\alpha}(t, a)\ \partial^\alpha \in \mathcal{P} \otimes \mathbb{C}[[t]] \otimes \boldsymbol{\mathcal{A}}\, ,\ u_{i\alpha}(t, a) \in \mathbb{C}[[t]] \otimes \boldsymbol{\mathcal{A}}. \tag{9.11}$$

Then there exists $K \in 1 + \mathcal{P}_- \otimes \mathbb{C}[[t]] \otimes \boldsymbol{\mathcal{A}}$ such that for $1 \leq i \leq N$,

$$L_i = K(\partial_i)K^{-1},$$
$$\partial_\alpha K = -(K(\partial^\alpha)K^{-1})_-\ K. \tag{9.12}$$

Equivalently, the operators $\{L_1, \ldots, L_N\}$ admit a wave function $\psi(x, t, a)$ satisfying:

$$L_i\psi = x_i \cdot \psi, \qquad 1 \leq i \leq N,$$
$$\partial_\alpha \psi = (L^\alpha)_+\psi, \qquad 0 \subset \alpha. \tag{9.13}$$

The next step is to show that there exists a Baker function corresponding to a point in $\mathrm{Gr}(A)$ that is indeed such a wave function, and one that can be implemented so as to achieve the conjugation into the algebra A, just as for the one-variable case. In order to see this, we introduce formal oscillating functions

$$\phi(t, x, a) = \left(1 + \sum_{0\subset\alpha} u_\alpha(t, a)\, x^\alpha\right) e^{\xi(t,x)}, \tag{9.14}$$

where

$$e^{\xi(t,x)} := \exp\left(\sum_{0\subset\beta} t_\beta\, x^{-\beta}\right), \qquad t = \{t_\beta : 0 \subset \beta\}, \tag{9.15}$$

and $u_\alpha(t, a) \in \mathbb{C}[[\{t_\alpha\}_{0\subset\alpha}]] \otimes \boldsymbol{\mathcal{A}}$.

Relative to $W \in \mathrm{Gr}(A)$ we define the function ψ_W and a basis $w = \{w_i\}_{i\in I} \in V(p, A)$, such that

$$\psi_W(t, x, a) = \sum_{i\in I} t_{v(w_i)} w_i, \tag{9.16}$$

where $v : A \longrightarrow \mathbb{Z}^N$ maps $\sum_\alpha u_\alpha\, x^\alpha$ to the multi-index α such that $u_\alpha = 0$, $\forall\beta < \alpha$.

Using the bilinear form on $V \otimes \boldsymbol{\mathcal{A}}$ in the data (**) 3., there is the following 'orthogonality' relation deduced from [29]. Relative to $W \in F_0 \subset \mathrm{Gr}(A)$ in (**) 2., we have for all s, t:

$$\langle \psi_W(t, x, a),\ \phi_W(s, x_N, a)\rangle\ = 0. \tag{9.17}$$

In particular, there exists a natural map from the set of such wave functions for the $KP(N)$ hierarchy to $F_0 \subset \mathrm{Gr}(V)$, and conversely, if $W \in F_0$, then the Baker function corresponding to W is a wave function for this hierarchy (these facts follow essentially from [29] Theorems 3.5 and 3.6). Thus we have established the desired properties of the function ψ_W which implements the conjugation of $\mathcal{P} \otimes \mathbb{C}[[t]] \otimes \boldsymbol{\mathcal{A}}$ into a subring of A. □

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